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Augustine H. Gray, Jr. (S'56-M'65-SM'79), for a photograph and biography, see p. 376 of the August 1980 issue of this TRANSACTIONS.

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On the Computation of the Complex Cepstrum

BIR BHANU AND JAMES H. MCCLELLAN

Abstract—**A technique based on fitting splines to the phase derivative curve is presented for the efficient and reliable computation of the complex cepstrum. The frequency sampling and the thresholds of the adaptive phase unwrapping algorithm are studied as a function of the radii of the zeros of the signal.**

I. INTRODUCTION

The main computational burden in computing the complex cepstrum is the determination of a continuous, odd, and periodic phase function [1] for the Fourier transform of the signal. One generally starts from samples of the phase modulo 2π (obtained via an inverse tangent routine) and then attempts to "unwrap" the phase. Phase unwrapping algorithms published by Schafer [1] and Tribolet [2] are available in the literature. Such algorithms encounter problems when zeros of the signal are clustered near the unit circle. In such cases the use of the trapezoidal rule for integrating the phase derivative in Tribolet's algorithm results in large truncation errors and many step interval adaptations are required. In this paper the use of a piecewise polynomial interpolation scheme known as cubic splines is presented. This method gives a more accurate rule of integration [3] and can be easily incorporated into Tribolet's phase unwrapping algorithm. A sensitivity analysis is also presented to determine the minimum frequency sampling step size when the zeros of the signal are located very close to the unit circle. Furthermore, the selection of the incremental and consistency thresholds of Tribolet's adaptive algorithm are studied.

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II. PHASE UNWRAPPING USING CUBIC SPLINES

The problem of phase unwrapping can be viewed as fitting a curve to a finite set of known values of the phase derivative and then performing numerical integration to obtain the unwrapped phase consistent with the known principal values of the phase. Consider the fitting of cubic splines $S(\omega)$, having continuous first and second derivatives, to the phase derivative between $\omega = o$ and $\omega = \pi$ at N given points. Let the unwrapped phase of the sequence $x[n]$ be denoted by $\arg [X(e^{j\omega})]$ and the first and second derivatives of the phase be $\arg'[X(e^{j\omega})]$ and $\arg''[X(e^{j\omega})]$, respectively. The values of the phase derivative are known at discrete points $\omega_i (1 \leqslant i \leqslant N)$. A set of cubics $S(\omega)$ is passed through the points $\arg'[X(e^{j\omega_i})]$ using a new cubic in each interval. It is required that the slopes and curvature be the same for the cubics that join at each point. Then the estimate of the unwrapped phase is obtained by integration [5]

$$\int_{\omega_i}^{\omega_{i+1}} S(\omega)\, d\omega = \frac{\Delta\omega_i}{2} [\arg'[X(e^{j\omega_{i+1}})] + \arg'[X(e^{j\omega_i})]] - \frac{\Delta\omega_i^2}{12} [S'_{i+1} - S'_i], \quad i = 1, 2, \cdots, N-1 \tag{1}$$

where

$$\Delta\omega_i = \omega_{i+1} - \omega_i \quad \text{and} \quad S'_i = S'(\omega_i)$$

with

$$S'_1 = \arg''[X(e^{j\omega_1})] \quad \text{and} \quad S'_N = \arg''[X(e^{j\omega_N})].$$

The phase first and second derivatives can be computed using three Fourier transforms. These equations are

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$$\arg'[X(e^{j\omega})] = \frac{X_R X_I' - X_I X_R'}{|X|^2} \tag{2}$$

$$\arg''[X(e^{j\omega})] = \frac{1}{|X|^4}\,[|X|^2\{X_R X_I'' - X_I X_R''\} + 2X_R X_I\{(X_R')^2 - (X_I')^2\} + 2X_R' X_I'\{X_I^2 - X_R^2\}] \tag{3}$$

where

$$X_R'(e^{j\omega}) + jX_I'(e^{j\omega}) = -j\,\mathrm{FT}\{nx[n]\} \tag{4}$$

$$X_R''(e^{j\omega}) + jX_I''(e^{j\omega}) = -\mathrm{FT}\{n^2 x[n]\}. \tag{5}$$

Note that the first term of (1) is the trapezoidal rule and the second term is similar to the truncation term usually found in the trapezoidal rule. The quantities S_i' in (1) can be determined either by efficiently solving a tridiagonal set of equations [5] or by computing (3). (S_i' is an approximation to $\arg''[X(e^{j\omega_i})]$.) However, the DFT approach is preferred, especially in those situations where the zeros of the sequence are close to the unit circle and the second derivative of phase must be known very accurately.

III. Computational Strategies and Analysis

A number of issues arise in connection with the use of cubic spline interpolation within Tribolet's adaptive phase unwrapping algorithm. These include the sensitivity of the phase unwrapping to zero location and the amount of computation required. The cubic spline modification requires the first and second derivatives of the phase at each DFT point. Thus three FFT's are now required instead of two as in Tribolet's implementation [2]. This increase in the amount of computation per frequency point should be offset by a reduction in the number of times that the algorithm must adapt. It has been observed with speech signals [5] that the spline integration may reduce the number of times that the algorithm had to adapt by about 20 percent. However, on the basis of overall computation time it does not provide a significant saving; the computation time required [5] is comparable to that of Tribolet's algorithm. The primary advantage of using this new modification is an increase in the reliability and precision of the phase unwrapping technique.

When the zeros of the signal are clustered near the unit circle, the accuracy of the adaptive integration becomes critical. Examples where Tribolet's algorithm fails but the cubic spline approach works can be constructed. Consider a sixth-order test signal with the following zeros:

$$Z_1, Z_1^* \quad \text{at} \quad 0.99999\, e^{\pm j\pi/4}$$

$$Z_2, Z_2^* \quad \text{at} \quad 1.00001\, e^{\pm j((\pi/4)+(\pi/8192))}$$

and

$$Z_3, Z_3^* \quad \text{at} \quad 0.99999\, e^{\pm j((\pi/4)+(\pi/4096))}. \tag{6}$$

When an FFT size of 1024 is employed, the trapezoidal rule for integration will not obtain the correct phase but the spline integration will succeed. (Note: the thresholds of Tribolet's algorithm were taken to be $\theta_i = 2$ and $\theta_c = 1$; see below.) For this test signal, Tribolet's algorithm [2] fails because the 1024 point FFT does not resolve the important peaks in the phase derivative. Fig. 1(a) and (b) show the first and second derivatives of the phase computed from a 1024 point FFT. The spline integration succeeds because the integration is improved through the use of second derivative information and, in fact, requires no adaptation to unwrap the phase of (6). Tribolet's algorithm would succeed only if the FFT size were increased significantly. The correct unwrapped phase is shown in Fig. 2.

The performance of either integration technique can be adversely affected by rounding errors. The use of double precision for calculating the DFT's of $x[n]$, $nx[n]$, $n^2x[n]$ is quite important when adaptation takes place because the phase derivative will be changing quite rapidly and thus very small errors in the DFT's can lead to very large errors in the integrated phase. As an aside, it has been found that Bonzanigo's [4] modification of Goertzel's algorithm for the computation of the DFT at a single frequency off the FFT grid has more roundoff noise than the direct DFT when considering zeros close to the unit circle [5].

The adaptive nature of Tribolet's algorithm [2] requires two thresholds: the so-called incremental and consistency thresholds. These parameters affect the computation time and reliability of the phase unwrapping algorithm. The relationship among these thresholds, the FFT size, and closeness of the zeros to the unit circle can be examined for cases where the zeros of the signal are known. The incremental threshold θ_i restricts the amount by which the phase may increase in one

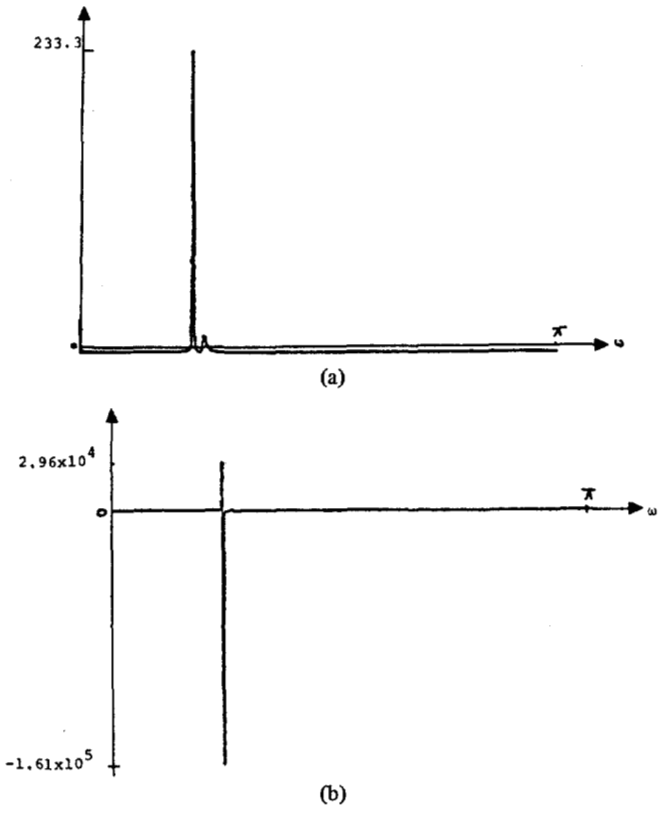

Fig. 1. (a) Phase first derivative for zero distribution of (6). Note that the figure was drawn based on 1024 point FFT and the sparse FFT sampling did not catch all the three spikes corresponding to three zero pairs. (b) Phase second derivative corresponding to (6).

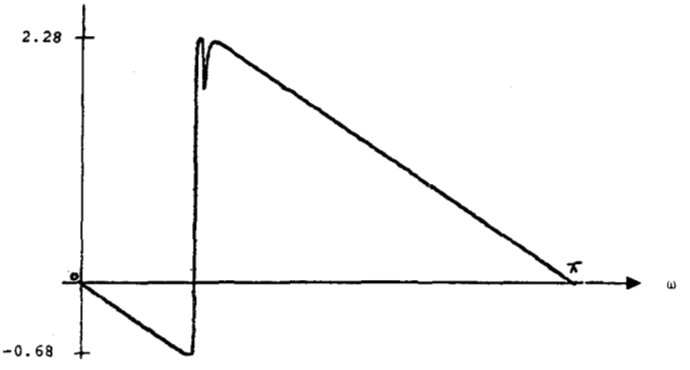

Fig. 2. Unwrapped phase after removal of linear phase for the example in (6).

TABLE I
SINGLE REAL ZERO AT $Z = a$, ($\theta_c = 0.65$)

a	$\Delta\omega_{max}$	Estimate of θ_i	$2\pi/\Delta\omega_{max}$	min FFT,N	Spline integrated phase at $\Delta\omega_{max}$	Unwrapped phase
0.95	0.18	3.42	34.9	64	1.836	1.203
0.99	0.035	3.46	179.5	256	1.896	1.273
0.995	0.017	3.38	369.5	512	1.863	1.275
0.999	0.0035	3.49	1795.1	2048	1.919	1.290
0.9995	0.0017	3.39	3695.9	4096	1.874	1.283
0.9999	0.00035	3.49	17951.9	32768	1.922	1.292
0.99995	0.00017	3.39	36959.9	65536	1.876	1.284
0.99999	0.000035	3.49	179519.5	262144	1.922	1.292
0.999995	0.000017	3.39	369599.1	524288	1.876	1.284
0.999999	0.0000035	3.49	1795195.8	2097152	1.922	1.292

integration step; the consistency threshold θ_c requires the integrated phase (modulo 2π) to be close to the known principal value of the phase.

The following experiment was carried out to study the interrelationship of θ_i, θ_c, transform length, and closeness of zeros to the unit circle. Consider a signal with one real zero at $Z = a$. As a approaches the unit circle the phase derivative contains a large spike at $\omega = 0$, where the phase changes rapidly. From the analytical expressions for the first and second derivatives of the phase, the behavior of the spline integration can be determined exactly near $\omega = 0$. Thus a value for θ_c is chosen and a maximum frequency increment $\Delta\omega_{max}$ is calculated under the constraint that the spline integrated phase give the true phase to within the amount θ_c. Table I was constructed by repeating this process for many values of the zero a near the unit circle.

The third column of Table I shows the product of the maximum value of phase derivative and $\Delta\omega_{max}$, which serves as a reasonable estimate of the incremental threshold θ_i. From this experiment it follows that the reasonable value of the incremental threshold to be used in conjunction with the consistency threshold of 0.65 is 3.5. The value of $\Delta\omega_{max}$ governs the "effective" DFT size to be used. This length is a combination of the initial FFT size and the number of adaptations permitted. For a second-order signal a similar analysis gave the value of $\theta_i = 2$ corresponding to $\theta_c = 1$ [5]. These threshold values have been used successfully in a number of higher order examples including the example of (6). Further extension of this work to higher order examples has been considered in detail in [5], which does not require the *a priori* distribution of zeros.

IV. CONCLUSION

The use of cubic spline integration improves the reliability of an adaptive phase unwrapping algorithm, especially when the zeros of the sequence are quite close to the unit circle. Analysis of first- and second-order examples shows that, for zeros very close to the unit circle, the minimum FFT size varies inversely with the distance to the unit circle. These ideas have also been applied for the computation of two-dimensional complex cepstrum and checking the stability of one- and two-dimensional recursive digital filters [5].

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A New ARMA Spectral Estimator

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Abstract—**Recently, several researchers have proposed methods for estimating an autoregressive-moving average power spectral density without the need to determine the moving average parameters. However, these techniques do not guarantee a nonnegative spectral estimate and thus sometimes lead to invalid estimates. A simple procedure is proposed for obtaining a nonnegative spectral estimate given an estimate of the autoregressive parameters.**

I. INTRODUCTION

The estimation of the parameters of an autoregressive-moving average (ARMA) process usually involves a spectral factorization to determine the moving average (MA) parameters [1]. However, if only the spectral estimate is desired, then several techniques are available which do not require a spectral factorization. These methods are now reviewed [2]-[4].

If X_t is an ARMA (p, q) process, then the power spectral density is given as

$$P_x(Z) = \frac{B(Z)\,B(Z^{-1})}{A(Z)\,A(Z^{-1})} \tag{1}$$

where

$$B(Z) = \sum_{k=0}^{q} b_k Z^{-k}$$

$$A(Z) = \sum_{k=0}^{p} a_k Z^{-k} \qquad a_0 \equiv 1.$$

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